Comparison of Implicitly Restarted Arnoldi Algorithm with Jacobi-Davidson Algorithm for the Calculation of Eigenvalues

Marcel Hundscheid, Robert Dimitrovski
Institute of Electrical Energy Systems
Friedrich-Alexander University Erlangen-Nuremberg (FAU)
Erlangen, Germany
marcel.hundscheid@fau.de, Robert.dimitrovski@fau.de

***Abstract*—The common used software tools for small signal stability analysis are using QR Algorithms as well as Arnoldi Algorithms to calculate the eigenvalues of the electrical power grids' system matrix. In the past decade, a new algorithm called Jacobi-Davidson algorithm for the calculation of eigenvalues was developed. Within this paper, the Jacobi-Davidson algorithm is benchmarked and compared to the implicitly restarted Arnoldi Algorithm. The benchmark revealed that Jacobi-Davidson Algorithm is an alternative algorithm for the calculation of eigenvalues in the context of small signal stability analysis.**

Keywords-component; small signal stability; eigenvalues; Jacobi-Davidson Algorithm

I. INTRODUCTION

Small signal stability is the ability of an electric power system to maintain synchronism while being subjected by small disturbances. The analysis of small signal stability examines whether a system is stable or not. This information can be found by calculating the eigenvalues of the state matrix. If the state matrix contains eigenvalues with positive real part the behavior of the system is instable. Furthermore, eigenvalues with a large imaginary part can lead to unstable oscillations.

The conventional QR algorithm calculates all the eigenvalues to examine the small signal stability of the power system. This method is inadequate for large system since computational time and storage rise as the system size becomes larger. However, this algorithm is highly effective for small systems. While investigating the small-signal stability, the eigenvalues containing the largest real parts are of greatest interest. To avoid calculation of all eigenvalues of the system matrix algorithms were developed in the past which can calculate eigenvalues with certain properties. Among them the most widely spread algorithm is the Implicitly Restarted Arnoldi Algorithm. This algorithm will be compared with the Jacobi-Davidson Algorithm as a new method which is not commonly used for small signal stability analysis. Both methods are subspace iteration methods [1, 5].

II. IMPLICITLY RESTARTED ARNOLDI ALGORITHM

The Implicitly Restarted Arnoldi Algorithm is suitable for the calculation of eigenvalues with certain qualities. It is a well-known method and it is used in many calculation software tools for small signal stability analysis of electrical power grids, like PowerFactory, PSS Netomac or NEPLAN. The implicitly restarted Arnoldi algorithm is an advanced method of the k-step Arnoldi algorithm, which is the basic method and is explained first.

A. *k-step Arnoldi Algorithm*

The k-step Arnoldi Algorithm is an iterative method to find k eigenvalues of an $n \times n$ matrix. Thereby is k much smaller than n. To find the eigenvalues of the matrix **A** the eigenvectors are approximated in the matrix **H**. The matrix **A** is manipulated to obtain an orthogonal matrix **V** and an upper Hessenberg matrix **H** [1, 2]. The k Arnoldi-steps are defined as [1]:

$$AV_k = V_{k+1} H_k \tag{1}$$

The matrix V includes the approximated eigenvectors and the diagonal entries of the upper Hessenberg matrix H are the closed up eigenvalues of the original matrix A [2].

B. *Exactness of cloesd up eigenvalues*

To get an idea how exact the closed up values are, (1) can be written as follows:

$$AV_k = (V_k , v_{k+1}) \begin{pmatrix} H_k \\ h_{k+1} e_k^T \end{pmatrix} = V_k H_k + h_{k+1,k} v_{k+1} e_k^T \tag{2}$$

If the element $h_{k+1,k}$ is equal zero than is the approximation exactly and the k basic vectors of **V** are linear independent [1]. That means that $\mathbf{H}_k$ has the same eigenvalues as **A**. In practice it can't be assumed to find zero exactly. Because of this a limiting value ε is introduced which can be chosen very small [3].

$$\left\|w_{k+1}\right\|_2 = h_{k+1,k} \le \varepsilon \left\|Av_{k-1}\right\|_2 \quad (3)$$

That means the smaller the element $h_{k+1,k}$ is the more accurate is the approximated solution of the eigenvalues.

C. Implicitly Restarted Arnoldi Algorithm

In the first stept of the Implicitly Restarted Arnoldi Algorithm *k* Arnoldi steps are calculated. After this *p* more Arnoldi steps are calculated. Equation (2) can now be written together with $k + p = m$ as follows [1, 2]:

$$AV_m = (V_m, v_{m+1}) \begin{pmatrix} H_m \\ h_{m+1} e_m^T \end{pmatrix} = V_m H_m + h_{m+1,m} v_{m+1} e_m^T \quad (4)$$

With (4) an Arnoldi basis is calculated which is larger than the originally wanted one. The additional information can be used to obtain eigenvalues with certain qualities. The eigenvalues of the small $\mathbf{H}_m$ Hessenberg matrix can easily be calculated with a QR-Algortihm and be sorted to certain qualities like:

- Largest / smallest real part
- Largest / smallest imaginary part
- Largest absolute length

Through sorting there is obtained a group of wanted and a group of unwanted eigenvalues. The unwanted eigenvalues are used as shifts within the QR-Shift-Algorithm through which the orthogonal **Q** Matrix is calculated. The **Q** matrix is used to update the **H** and **V** matrices [1, 2]:

$$\hat{V}_m = V_m Q \quad (5)$$

$$\hat{H}_m = Q^* H_m Q \quad (6)$$

With this transformation the information of the additional *p* shifts are included in the *k* rows [1]. The algorithm can be restarted with a new starting vector which is orthogonal to the basic vectors of **V** to find the additional eigenvalues [1, 4].

III. Jacobi-Davidson Algorithm

The Jacobi-Davidson Algorithm is like the Arnoldi Algorithm suitable for the calculation of eigenvalues with certain qualities. For the small signal stability analysis, the eigenvalues with the largest real part are the most interesting ones. Following Jacobi method, the idea is to project the problem to an orthogonal vector subspace which is defined: [5]

$$B = \left(I - u_k u_k^*\right) A \left(I - u_k u_k^*\right) \quad (7)$$

Jacobi method is extended with the Davidson method. The goal is to find an eigenpair (Θ, u) which is a good approximation of an eigenvector and the related eigenvalue (λ, x) of **A**. An eigenpair can be calculated with the Rayleigh-Ritz method [1]:

$$\Theta_k = u_k^* A u_k \quad (8)$$

Thereby is Θ the Ritzvalue and $\boldsymbol{u}$ is the Ritzvector. The vector subspace $U = (u_1, ..., u_k)$ is to be extended in the way that the extension is orthogonal. According to [5] is defined $v := t$ with $u \perp t$. This should full fill the eigenvalue equation:

$$A(u_k + t) = \lambda(u_k + t) \quad (9)$$

Vector t can be calculated with the correction equation for which the residuum r is necessary and which can be calculated according to the following equation [6]:

$$-r_k = (A - \Theta_k I) u_k \quad (10)$$

The residuum is used to get an idea how accurate the calculation is. The more accurate the approximation of the Ritzvalue and the Ritzvector towards the eigenvalue and the eigenvector the smaller is the value of the residuum.

$$\left(I - u_k u_k^*\right)\left(A - \Theta_k I\right)\left(I - u_k u_k^*\right)t = -r_k \quad (11)$$

With the correction equation which is the most important equation of the Jacobi-Davidson method the vector **t** is calculated. In practise this is done with powerful calculation algorithms for example like generalized minimal residual method (GMRES).

IV. Benchmark System and Test Matrices

To benchmark both algorithms the eigenvalues of the IEEE 50 Generator system [7] (Figure 1) are calculated and the time needed for the calculation is measured.

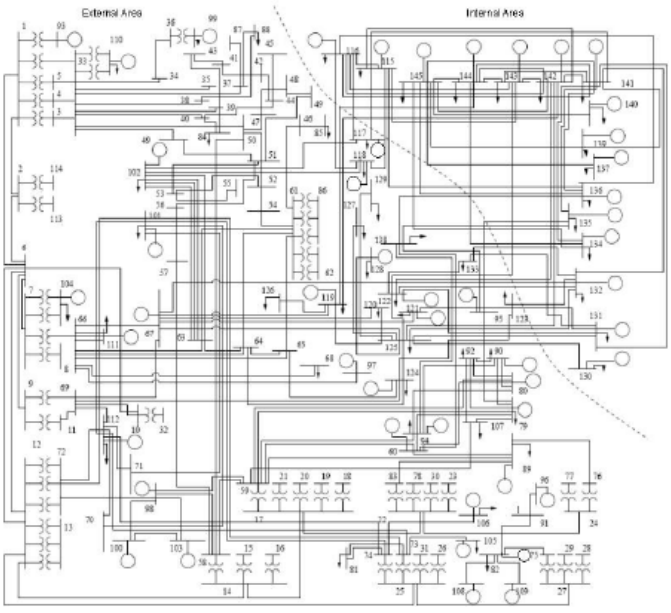

Figure 1. IEEE 50 Generator Test Case [7]

For further investigations of the calculation speed several test matrices were constructed in Matlab®. These test matrices

have similar qualities as typical system matrices of electrical power systems, such as:

- Asymmetric
- Sparse
- Real values

The construction was done by the Matlab-code:

$$G = numgrid('N',34)$$
$$B = delsq(G)$$
$$A = sprandn(B) + i \bullet sprandn(B)$$

which leads to sparse, asymmetric matrices with complex entries. The complex entries make the calculation of the eigenvalues even more difficult. The argument 34 in the function "numgrid" of the Matlab-code leads to a 1024 × 1024 matrix with the qualities listed above. In the same way, a second test matrix with a scope of 2025 × 2025 was generated to verify the performance of the algorithms for even larger matrices.

V. Comparison of Jacobi-Davidson Algorithm with Implicitly Restarted Arnoldi Algorithm

Both algorithms were implemented into Matlab and the eigenvalues of the system matrix with the largest real part were calculated. Thereby the time was measured which was needed by the algorithms to calculate the ten eigenvalues with the largest real part. To prove the correctness of the calculation the results were compared with the results of the Matlab function "eig".

In the first case the eigenvalues of the system matrix of the 50 knots test grid were calculated and the time the calculation took was measured. The accuracy for the calculations of A, B, C was set to 10^-8.

A. *Calulation time of the 50 generator test grid*

The calculation of the eigenvalues for the 50 generator test grid with the Implicitly Restarted Arnoldi Algorithm is quicker than the calculation of the Jacobi-Davidson algorithm.

TABLE 1. CALULATION TIME OF THE 50 GENERATOR TEST GRID

Arnoldi [s]	JD [s]
0.4991	2.2576

B. *Calulation time of the 1024 × 1024 test matrix*

Table lists the calculation time of the 1024 × 1024 test matrix of Implicitly Restarted Arnoldi Algorithm (IRAM) and Jacobi-Davidson Algorithm. To calculate the ten eigenvalues with the largest real part the IRAM is quicker than Jacobi-Davidson Algorithm.

TABLE 2. CALULATION TIME OF THE 1024 × 1024 TEST MATRIX

Arnoldi [s]	JD [s]
1.2751	2.0133

C. *Calulation time of the 2025 × 2025 test matrix*

Table lists the calculation time of the 2025 × 2025 test matrix of Implicitly Restarted Arnoldi Algorithm and Jacobi-Davidson Algorithm. To calculate the ten eigenvalues with the largest real part the IRAM is quicker than Jacobi-Davidson Algorithm.

TABLE 3. CALULATION TIME OF THE 2025 × 2025 TEST MATRIX

Arnoldi [s]	JD [s]
1.3388	2.6814

D. *Influce of the accuracy on the calculation time*

To benchmark the influence of the accuracy of the calculation the ten eigenvalues with the largest real part of the 1024 × 1024 test matrix with different tolerance values. The tolerance is the difference from the result of the Algorithm of choice towards the result of the matlab function "eig", which is supposed to deliver the correct result.

TABLE 4. INFLUCE OF THE ACCURACY ON THE CALCULATION TIME

Calculation tolerance	Arnoldi [s]	JD [s]
10^-4	0,4114	0,4607
10^-6	0,5535	0,6529
10^-8	1,2469	1,3405
10^-10	1,5586	2,2238

The calculation took more time if the accuracy is more precisely, which was to be expected.

Conclusion

With the Jacobi-Davidson Algorithm a new method for the calculation of eigenvalues in the context of small signal stability analysis is introduced. It calculates successfully the eigenvalues of system matrices of electrical power grids as well as benchmark matrices and can be seen as an alternative to the established methods. With further development the calculation time can be improved to challenge the performance of the Implicitly Restarted Arnoldi Algorithm.

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